Now consider the case when $a = b = c > 0$. The inequality of the problem is then equivalent to

$$a\left(\frac{2\sin x}{x} + \frac{\tan x}{x} - 3\right) > 0.$$

We have

$$\frac{2\sin x}{x} = 2 - \frac{x^2}{3} + \frac{x^4}{60} - \ldots \text{ and}$$

$$\frac{\tan x}{x} = 1 + \frac{x^2}{3} + \frac{2x^4}{15} + \frac{17x^6}{315} + \ldots.$$

Then the left side of the inequality is

$$a\left[\left(2 - \frac{x^2}{3} + \frac{x^4}{60} - \ldots\right) + \left(1 + \frac{x^2}{3} + \frac{2x^4}{15} + \frac{17x^6}{315} + \ldots\right) - 3\right],$$

and the inequality of the problem can be written as

$$a\left[\left(\frac{x^4}{60} - \frac{2x^6}{7!} + \ldots\right) + \left(\frac{2x^4}{15} + \frac{17x^6}{315} + \ldots\right)\right] > 0.$$

On the interval $\left(0, \frac{\pi}{2}\right)$, the alternating series part is a convergent series whose terms in absolute value are decreasing and whose first term is positive. Thus both series inside the brackets are positive and the inequality of the problem is correct for positive numbers a, b, and c for x in the interval $\left(0, \frac{\pi}{2}\right)$.

Also solved by Arkady Alt, San Jose, CA; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata," University,Rome, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Boris Rays, Brooklyn, NY, and the proposer.

- **5233:** *Proposed by Anastasios Kotronis, Athens, Greece*

Let $x \geq \frac{1 + \ln 2}{2}$ and let $f(x)$ be the function defined by the relations:

$$f^2(x) - \ln f(x) = x$$
$$f(x) \geq \frac{\sqrt{2}}{2}.$$

- 1. Calculate $\lim_{x \to +\infty} \frac{f(x)}{\sqrt{x}}$, if it exists.
- 2. Find the values of $\alpha \in \Re$ for which the series $\sum_{k=1}^{+\infty} k^\alpha \left(f(k) - \sqrt{k}\right)$ converges.
- 3. Calculate $\lim_{x \to +\infty} \frac{\sqrt{x} f(x) - x}{\ln x}$, if it exists.

Solution 1 by Arkady Alt, San Jose, CA

1. Since $x \geq \dfrac{1+\ln 2}{2}$ and $f(x) \geq \dfrac{\sqrt{2}}{2}$ then

$\ln f(x) + x \geq x + \ln\left(\dfrac{\sqrt{2}}{2}\right) \geq \dfrac{1+\ln 2}{2} - \dfrac{\ln 2}{2} = \dfrac{1}{2}$

and, therefore, for such x and $f(x)$ we have

$$f^2(x) - \ln f(x) = x \iff$$

$$f(x) = \sqrt{x + \ln f(x)} \text{ and}$$

$$f(x) \geq \sqrt{x + \ln\left(\frac{\sqrt{2}}{2}\right)} = \sqrt{x - \frac{\ln 2}{2}}.$$

$$\text{Hence, } \lim_{x\to+\infty} f(x) = \infty$$

Since $f(x) > 0$ then

$$f^2(x) - \ln f(x) = x \iff f(x) = \frac{x}{f(x)} + \frac{\ln f(x)}{f(x)}$$

and, therefore,

$$f(x) \leq \frac{x}{\sqrt{x - \dfrac{\ln 2}{2}}} + \frac{\ln f(x)}{f(x)}.$$

Hence,

$$\frac{\sqrt{x - \dfrac{\ln 2}{2}}}{\sqrt{x}} \leq \frac{f(x)}{\sqrt{x}} \leq \frac{\sqrt{x}}{\sqrt{x - \dfrac{\ln 2}{2}}} + \frac{\ln f(x)}{\sqrt{x} f(x)}.$$

Since

$$\lim_{x\to+\infty} \frac{\sqrt{x - \dfrac{\ln 2}{2}}}{\sqrt{x}} = 1, \lim_{x\to+\infty} \frac{\sqrt{x}}{\sqrt{x - \dfrac{\ln 2}{2}}} = 1 \text{ and } \lim_{x\to+\infty} \frac{\ln f(x)}{f(x)} = 0.$$

Then by the squeeze principle we obtain

$$\lim_{x\to+\infty} \frac{f(x)}{\sqrt{x}} = 1.$$

2. First note that series $\sum\limits_{n=1}^{\infty} \dfrac{\ln n}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

(Let $p > 1$ and $\varepsilon = \dfrac{1-p}{2}$. Since $p - \varepsilon = \frac{3p-1}{2} > 1$ then series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{p-\varepsilon}}$ is convergent.

There is $n_0 \in N$ such that $\ln n < n^{\varepsilon}$ for all $n > n_0$ (because $\lim\limits_{n\to\infty} \dfrac{\ln n}{n^q} = 0$ for any $q > 0$).

Hence,

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^p} = \sum_{k=1}^{n_0} \frac{\ln k}{k^p} + \sum_{n=n_0+1}^{\infty} \frac{\ln n}{n^p} < \sum_{k=1}^{n_0} \frac{\ln k}{k^p} + \sum_{n=n_0+1}^{\infty} \frac{n^{\varepsilon}}{n^p} = \sum_{k=1}^{n_0} \ln k k^p + \sum_{n=n_0+1}^{\infty} \frac{1}{n^{p-\varepsilon}}.$$

If $p \leq 1$ then $\sum\limits_{n=3}^{\infty} \dfrac{\ln n}{n^p} > \sum\limits_{n=3}^{\infty} \dfrac{1}{n^p}$, where by p test $\sum\limits_{n=3}^{\infty} \dfrac{1}{n^p}$ is divergent series and, therefore, the series $\sum\limits_{n=1}^{\infty} \dfrac{\ln n}{n^p}$ is divergent.)

Also note that $\lim\limits_{x \to +\infty} \dfrac{\ln f(x)}{\ln x} = \dfrac{1}{2}$. Indeed,

$$\begin{aligned}
\lim_{x \to +\infty} \left(\frac{2 \ln f(x)}{\ln x} - 1 \right) &= 2 \lim_{x \to +\infty} \frac{\ln \left(\frac{f(x)}{\sqrt{x}} \right)}{\ln x} \\
&= 2 \lim_{x \to +\infty} \cdot \frac{1}{\ln x} \cdot \lim_{x \to +\infty} \ln \left(\frac{f(x)}{\sqrt{x}} \right) \\
&= 2 \lim_{x \to +\infty} \frac{1}{\ln x} \cdot \ln \left(\lim_{x \to +\infty} \frac{f(x)}{\sqrt{x}} \right) = 2 \cdot 0 \cdot \ln 1 = 0.
\end{aligned}$$

Since

$$f^2(x) - \ln f(x) = x \iff f(x) - \sqrt{x} = \frac{\ln f(x)}{f(x) + \sqrt{x}}, \text{ then}$$

$n^{\alpha} (f(n) - \sqrt{n}) = \dfrac{n^{\alpha} \ln f(n)}{f(n) + \sqrt{n}}$ and, therefore,

$$\lim_{n \to \infty} \frac{n^{\alpha} (f(n) - \sqrt{n})}{n^{\alpha - 1/2} \ln n} = \lim_{n \to \infty} \left(\frac{1}{n^{\alpha - 1/2} \ln n} \cdot \frac{n^{\alpha} \ln f(n)}{f(n) + \sqrt{n}} \right) = \lim_{n \to \infty} \frac{\sqrt{n} \ln f(n)}{(f(n) + \sqrt{n}) \ln n}$$

$$= \lim_{n \to \infty} \frac{\ln f(n)}{\left(\frac{f(n)}{\sqrt{n}} + 1 \right) \ln n} = \lim_{n \to \infty} \frac{\ln f(n)}{\ln n} \cdot \lim_{n \to \infty} \frac{1}{\left(\frac{f(n)}{\sqrt{n}} + 1 \right)} = \frac{1}{4}.$$

Thus, by the limit convergency test, both series $\sum\limits_{n=1}^{\infty} n^{\alpha} (f(n) - \sqrt{n})$ and $\sum\limits_{n=1}^{\infty} \dfrac{\ln n}{n^{1/2 - \alpha}}$ have the same character of convergency.

Since $\sum\limits_{n=1}^{\infty} \dfrac{\ln n}{n^{1/2-\alpha}}$ converges if $1/2 - \alpha > 1 \iff \alpha < -1/2$ and diverges if $1/2 - \alpha \leq 1 \iff -1/2 \leq \alpha$ we may conclude that series $\sum\limits_{n=1}^{\infty} n^{\alpha} (f(n) - \sqrt{n})$ is convergent if $\alpha < -1/2$ and divergent if $-1/2 \leq \alpha$.

3. Since

$$\begin{aligned}
f(x) - \sqrt{x} &= \frac{\ln f(x)}{f(x) + \sqrt{x}} \text{ then} \\
\lim_{x\to+\infty} \frac{\sqrt{x}f(x) - x}{\ln x} &= \lim_{x\to+\infty} \frac{\sqrt{x}\,(f(x) - \sqrt{x})}{\ln x} \\
&= \lim_{x\to+\infty} \frac{\sqrt{x}\ln f(x)}{\ln x\,(f(x) + \sqrt{x})} \\
&= \lim_{x\to+\infty} \frac{\sqrt{x}\ln f(x)}{\ln x\,(f(x) + \sqrt{x})} \\
&= \lim_{x\to+\infty} \frac{\ln f(x)}{\ln x} \cdot \lim_{x\to+\infty} \frac{1}{\dfrac{f(x)}{\sqrt{x}} + 1} = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}.
\end{aligned}$$

Solution 2 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

• 1. The function $t^2 - \ln t$ is strictly increasing for $t \geq 1/\sqrt{2}$ thus the equation $t^2 - \ln t = x$ admits a unique solution for any $x \geq (1 + \ln 2)/2$. This defines the function $f(x)$ of the problem which is strictly increasing and then it admits the limit L which can be finite of infinite. If L is finite the equation $f^2(x) = \ln f(x) + x$ cannot hold thus $L = +\infty$. Moreover the differentiability of $t^2 - \ln t$ assures the differentiability of $f(x)$ and in particular

$$2ff' = \frac{f'}{f} + 1 \implies f'(x) = \frac{f}{2f^2 - 1}$$

whence using l'Höpital

$$\lim_{x\to\infty} \frac{f^2}{x} = \lim_{x\to\infty} 2ff' = \lim_{x\to\infty} 2f\frac{f}{2f^2 - 1} = 1 \implies \lim_{x\to\infty} \frac{f}{\sqrt{x}} = 1$$

• 2. We have $f(x) = \sqrt{x} + o(\sqrt{x})$ thus $\ln f(x) = \frac{1}{2}\ln x + \ln(1 + o(1)) = \frac{1}{2}\ln x + o(1)$ and

$$f(x) = \sqrt{x + \ln f} = \sqrt{x + \frac{1}{2}\ln x + o(1)} = \sqrt{x}\sqrt{1 + \frac{1}{2}\frac{\ln x}{x} + \frac{o(1)}{x}}$$

whence

$$f(x) = \sqrt{x}\left(1 + \frac{1}{4}\frac{\ln x}{x} + o\left(\frac{\ln x}{x}\right)\right)$$

and then

$$\sum_{k=1}^{\infty} k^{\alpha}(f(k) - \sqrt{k}) = \sum_{k=1}^{\infty}\left[k^{\alpha-\frac{1}{2}}\frac{\ln k}{4} + k^{\alpha+\frac{1}{2}}o\left(\frac{\ln k}{k}\right)\right] = \sum_{k=1}^{\infty} k^{\alpha-\frac{1}{2}}\frac{\ln k}{4}\left(1 + o\left(\frac{1}{k}\right)\right).$$